NONLINEAR MAGNETOELASTIC WAVES

(NELINEINYE MAGNITOUPRUGIE VOLNY)

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L.A. POSPELOV
(Kharkov)

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Let us consider the nonlinearity effects on the propagation of elastic waves in a solid conductor in the presence of a magnetic field. Let us assume that the effects of nonlinearity and dissipation are weak at a distance whose order of magnitude is one wave length. Disregarding the cubic terms, the initial system of equations (1) reduces to one equation (15), whose solution is known [1]. It is shown that the effects of nonlinearity may lead to the formation of a discontinuity.

The effects of nonlinearity in the equations of motion and of dissipation in a medium on the propagation of waves represents a field of great interest and a great many papers are devoted to this subject. The investigation of the propagation of acoustic waves in a gas [1,2], electromagnetic waves in ferrite [3], and in a conducting liquid [4], and elastic waves in an isotropic solid [5], shows that as a consequence of interaction of the waves, the harmonic content of the original signal is greatly increased. This may lead to a distortion of the wave front and even to a discontinuity if the medium has no dispersion. When dispersion is present, the leading harmonics absorb more strongly and the tendency to form a discontinuity decreases. One must also expect all these effects in the propagation of magnetoelastic waves in an isotropic solid. The present paper is devoted to this subject.

Let us consider the system of self-consistent equations for a magnetic field H, and a vector displacement u of the medium [6]

$$\frac{\partial^2 u_i}{dt^2} = \frac{\partial \sigma_{ik}}{\partial x_k} + \frac{1}{4\pi\rho} [\text{rot } \mathbf{H} \cdot \mathbf{H}]_i, \qquad \text{div } \mathbf{H} = 0$$

$$\frac{\partial H_i}{\partial t} = \text{rot}_i \left[\frac{du}{dt} \cdot \text{rot } \mathbf{H}\right] + \frac{c^2}{4\pi\sigma} \Delta H_i \qquad \left(\sigma_{ik} = \frac{\partial E}{\partial (\partial u_i / \partial x_k)}\right) \qquad (1)$$

Here, σ_{ik} is the elastic stress tensor, E is the energy of elastic deformation

$$E = \frac{\mu}{4}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^2 + \left(\frac{k}{2} - \frac{\mu}{3}\right)\left(\frac{\partial u_e}{\partial x_e}\right)^2 + \left(\mu + \frac{A}{4}\right)\frac{\partial u_i}{\partial x_k}\frac{\partial u_e}{\partial x_i}\frac{\partial u_e}{\partial x_k} +$$
$$+ \left(\frac{B+k}{2} - \frac{\mu}{3}\right)\frac{\partial u_e}{\partial x_e}\left(\frac{\partial u_i}{\partial x_k}\right)^2 + \frac{A}{12}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_e}\frac{\partial u_e}{\partial x_i} + \frac{B}{2}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_e}{\partial x_e} + \frac{C}{3}\left(\frac{\partial u_e}{\partial x_e}\right)^3$$

Let us consider a solid occupying the half-space $x > 0$, and let a progressive wave be propagated along the x-direction (i.e. the wave vector k is parallel to the x-axis). Let us assume that an external constant magnetic field $\mathbf{H}_0$ lies in the xy-plane. Considering the original system of equations in component form, we obtain the result that the equations for u_z, H_z and H_x contain no other components. This means that if they are equal to zero at the boundary then they will also remain so throughout space. Thus we have

$$u_z = 0, \qquad H_z = 0, \qquad H_x = H_{0x} \tag{2}$$

Equations for the remaining components are

$$\begin{gathered}
\frac{d^2u_x}{dt^2} - c_e^2\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{4\pi\rho}H_{0y}\frac{\partial h_y'}{\partial x} = -\frac{1}{4\pi\rho}h_y'\frac{\partial h_y'}{dx} + \frac{2d}{\rho}\frac{\partial u_x}{\partial x}\frac{\partial^2 u_y}{\partial x^2} \\
\frac{d^2u_y}{dt^2} - c_t^2\frac{\partial^2 u_y}{\partial x^2} - \frac{1}{4\pi\rho}H_{0x}\frac{\partial h_y'}{\partial x} = \frac{d}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x}\right) \\
\frac{\partial h_y'}{\partial t} + H_{0y}\frac{\partial^2 u_x}{\partial t\,\partial x} - H_{0x}\frac{\partial^2 u_y}{\partial t\,\partial x} = -\frac{\partial}{\partial x}\left(h_y'\frac{\partial u_x}{\partial t}\right) + \frac{c^2}{4\pi\sigma}\frac{\partial^2 h_x'}{\partial x^2}
\end{gathered} \tag{3}$$

Where

$$h_y' = H_y - H_{0y}, \quad d = {}^1/_4 A + {}^1/_2(B+k) + {}^2/_3\mu, \quad q = {}^2/_3\rho c^2 + A + 3B + C$$

For a linear approximation (and with $\sigma = \infty$), let us assume a progressive wave with a phase velocity $u = k/\omega$, is propagated along the x-axis, which satisfies the dispersion equation

$$D_0(u) = \begin{vmatrix} 1-\beta_e^2 & 0 & \beta_a^2 \\ 0 & 1-\beta_t^2 & -\alpha\beta_a^2 \\ -1 & -1 & \alpha \end{vmatrix} = 0 \tag{4}$$

Where

$$\beta_e^2 = \frac{c_e^2}{u^2}, \qquad \beta_t^2 = \frac{c_t^2}{u^2}, \qquad \beta_a^2 = \frac{u_a^2}{u^2}, \qquad u_a^2 = \frac{H_{0y}^2}{4\pi\beta}, \qquad \alpha = \frac{H_{0x}}{H_{0y}}$$

In other words, for a linear approximation, all unknown values are functions of one independent variable $\psi = kx - \omega t$. For computation of the infinitesimal nonlinear and dissipation terms, one must assume [1, 2,4] that the unknown quantities depend on two variables: the complete phase $\psi = kx - \omega t$ and the "slow coordinate" $\xi = \varepsilon kx$, where $\varepsilon \ll 1$. The requirement that ε be infinitesimal reflects the fact that nonlinear and dissipation forces are weakly in evidence at a distance whose order of magnitude is one wavelength.*

Taking into account this observation and transferring from the variables (x, t) to the variables (ψ, ξ), we rewrite equation (3) in the form

$$\begin{gathered}(1-\beta_t^2)\frac{\partial^2 w}{\partial\psi^2} - \alpha\beta_a^2\frac{\partial h}{\partial\psi} = 2\varepsilon\beta_t^2\frac{\partial^2 w}{\partial\xi\,\partial\psi} + \varepsilon\alpha\beta_a^2\frac{\partial h}{\partial\psi} + \\ + \frac{2d}{u^2}\gamma\frac{\partial}{\partial\psi}\left(\frac{\partial v}{\partial\psi}\frac{\partial w}{\partial\psi}\right) - \gamma\frac{\partial^2 v}{\partial\psi^2}\frac{\partial w}{\partial\psi} + 2\gamma\frac{\partial v}{\partial\psi}\frac{\partial^2 w}{\partial\psi^2} \\ \frac{\partial h}{\partial\psi} - \frac{\partial^2 v}{\partial\psi^2} + \alpha\frac{\partial^2 w}{\partial\psi^2} = \varepsilon\frac{\partial^2 v}{\partial\psi^2} - \varepsilon\alpha\frac{\partial^2 w}{\partial\psi\,\partial\xi} + \gamma\frac{\partial}{\partial\psi}\left(h\frac{\partial v}{\partial\psi}\right) + R_m\frac{\partial^2 h}{\partial\psi^2} \\ (1-\beta_e^2)\frac{\partial^2 v}{\partial\psi^2} + \beta_a^2\frac{\partial h}{\partial\psi} = 2\varepsilon\beta_e^2\frac{\partial^2 v}{\partial\psi\,\partial\xi} - \varepsilon\beta_a^2\gamma h\frac{\partial h}{\partial\psi} + \frac{2q}{u^2}\gamma\frac{\partial v}{\partial\psi}\frac{\partial^2 v}{\partial\psi^2} + \frac{2d}{u^2}\gamma\frac{\partial w}{\partial\psi}\frac{\partial^2 w}{\partial\psi^2}\end{gathered} \tag{5}$$

Where

$$h = \frac{h'}{H_{0y}\gamma}, \qquad v = \frac{u_x}{u}, \qquad w = \frac{u_y}{u}, \qquad \gamma = ku_0, \qquad R_m = \frac{c^2\omega}{4\pi\sigma u^2}$$

In passing from system (3) to equation (5) use is made of the relationship

$$\varepsilon \sim \gamma + R_m \ll 1 \tag{6}$$

as well as the omitting of terms containing ε^3.

Further, it is found convenient to introduce new unknown functions having the following relationships:

$$V = -\frac{1-\beta_t^2}{\alpha(1-\beta_e^2)}v, \qquad W = w, \qquad h = \frac{1-\beta_t^2}{\alpha\beta_a^2}h \tag{7}$$

* The idea of this method essentially coincides with the idea of the Krylov-Bogoliubov-Mitropol'skii (KBML method, [7] and the simplified equations obtained in (5) by introducing the "slow coordinate" may be considered as the initial point in using the KBM method. The KBM method is not employed here, since the exact solution for equation (15) is known.

and to transform the system of equations into the following

$$\frac{\partial^2 V}{\partial\psi^2}-\frac{\partial^2 W}{\partial\psi^2}=\Phi_1, \qquad \frac{\partial^2 W}{\partial\psi^2}+\frac{\partial h}{\partial\psi}=\Phi_2, \qquad \frac{\partial^2 V}{\partial\psi^2}-\frac{\partial h}{\partial\psi}=\Phi_3 \tag{8}$$

Where

$$\Phi_1=\frac{\alpha}{1-\beta_t^2}\left\{2\varepsilon\beta_e^2\, v_{\xi\psi}-\varepsilon\beta_a h_\xi-\beta_a\gamma h h_\psi+\frac{2q}{u^2}\gamma v_\psi v_{\psi\psi}\right\}$$

$$\Phi_2=\frac{1}{1-\beta_t^2}\left\{2\varepsilon\beta_t^2 w_{\xi\psi}+\varepsilon\alpha\beta_a h_\xi+\frac{2d}{u^2}\partial\,(v_\psi w_\psi)_\psi-\gamma v_{\psi\psi}w_\psi+2\gamma v_\psi w_{\psi\psi}\right\} \tag{9}$$

$$\Phi_3=\frac{1}{\alpha}\left\{\varepsilon v_{\xi\psi}-\alpha\varepsilon w_{\xi\psi}+\gamma\,(h v_\psi)_\psi+R_m h_{\psi\psi}\right\}$$

The subscripts ψ and ξ of the unknown functions mean differentiation with respect to ψ and ξ, respectively.

The system of equations written in the form of (8) will be called the reduced system of equations. Further, it is verified that to write the initial system of equations in the form indicated will be essential for the application the method. By this formulation, the method becomes applicable to other systems of equations which may contain a larger number of initial equations.

Such a system of self-consistent equations in magnetohydrodynamics for a magnetic field H, vector velocity u, density ρ, temperature T and entropy S also reduces to the following system of four reduced equations (for two components of velocity, one component of the magnetic field, and density):

$$\frac{\partial x_1}{\partial\psi}-\frac{\partial x_2}{\partial\psi}=\Phi_1, \qquad \frac{\partial x_3}{\partial\psi}-\frac{\partial x_4}{\partial\psi}=\Phi_3, \qquad \frac{\partial x_2}{\partial\psi}-\frac{\partial x_3}{\partial\psi}=\Phi_2, \qquad \frac{\partial x_4}{\partial\psi}-\frac{\partial x_1}{\partial\psi}=\Phi_4 \tag{10}$$

where Φ_i is a linear combination of terms with the form

$$(x_m, x_n)_\psi, \qquad x_{m\psi\psi} \qquad (i, m, n = 1, 2, 3, 4)$$

The equation describing the propagation of an acoustic wave in a heat conducting viscous gas [2] reduces to a system of two reduced equations for velocity and density

$$\frac{\partial x_1}{\partial\psi}-\frac{\partial x_2}{\partial\psi}=\Phi_1, \qquad \frac{\partial x_2}{\partial\psi}-\frac{\partial x_1}{\partial\psi}=\Phi_2 \tag{11}$$

Let us consider equation (8) as algebraic, and, solving for their relative values

$$x_1=V_\psi, \qquad x_2=W_\psi, \qquad x_3=h_\psi$$

we obtain

$$d_0 x_i = d_i \qquad (i = 1, 2, 3), \qquad d_0 = \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix} \tag{12}$$

Here d_i is a determinant obtained by replacing the ith column in d_0 by a column composed of absolute terms of equation (8). Using equations (9) and (10), it is easy to show that

$$d_1 = d_2 = d_3 = \Phi_1 - \Phi_2 + \Phi_3 = 0 \tag{13}$$

The following equalities are obtained from the system of equations (8) with an accuracy extending up to terms of order ε^2:

$$V_\psi = W_\psi = h = \eta \tag{14}$$

Substituting expression (14) into equation (13), we obtain an equation which satisfies the characteristic function

$$\eta_\varsigma + \eta\eta_\psi = \delta\eta_{\psi\psi} \tag{15}$$

Where

$$\zeta = \frac{b}{a}\xi, \qquad \delta = \frac{c}{b}, \qquad a = -\varepsilon\left[\frac{1+\beta_e^2}{1-\beta_e^2} + \frac{(1+\beta_t^2)^2}{1-\beta_t^2} + \frac{1-\beta_t^2}{\alpha^2(1-\beta_e^2)^2} + 1\right]$$

$$b = \frac{\alpha\gamma}{1-\beta_t^2}\left[\frac{2d}{u^2} + \frac{(1-\beta_t^2)}{\alpha^2}\left(\frac{2(q-3)}{u^2(1-\beta_e^2)} - \frac{1}{\beta_a^2}\right)\right] +$$

$$+ \frac{\gamma}{\alpha(1-\beta_e^2)}\left(\frac{4d}{u^2}+1\right) - \frac{\gamma(1-\beta_t^2)^2}{\alpha^3\beta_a^2(1-\beta_e^2)}, \qquad c = \frac{R_m(1-\beta_t^2)}{\alpha\beta_a^2}$$

The transformations used here are valid if the following expressions hold:

$$ku_0 + \frac{c^2\omega}{4\pi u^2\sigma} \ll 1, \qquad \xi \lesssim 1 \tag{16}$$

The second of these expressions has no limitation since the deformation process of the wave front takes place within the interval $\xi \lesssim 1$ ($x \lesssim \varepsilon^{-1}$), that is, equation (15) completely describes the entire process of nonlinear deformation of the wave front. Let us note that expression (14) represents the solution of the problem under discussion, differing from the exact one by a third order infinitesimal ε^3. Taking into account that the second order terms ε^2 in the resulting solution have no meaning, since neglecting third order terms ε^3 in the equation determining the derivative of the solution with respect to the slow coordinate for $\xi \sim 1$, in the solution a second order error* ε^2 is

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* The situation here is similar to that which is met in applying the KBM method to ordinary differential equations (see, for example, [7, p.42]).

introduced. In the dimensions of the quantities obtained the solutions have the form

$$u_{x,\psi} = -u_0 \frac{1-\beta_t^2}{\alpha(1-\beta_e^2)}, \quad \eta, \quad u_{y,\psi} = u_0\eta, \quad h_y 1 = H_{0y}\gamma \frac{1-\beta_t^2}{\alpha\beta_a^2}\eta \tag{17}$$

The solution of equation (15) may be written [1] in the form

$$\eta = -2\delta \frac{\theta_\psi}{\theta} \tag{18}$$

where $\theta = \theta(\zeta, \psi)$ satisfies the equation of heat conduction

$$\frac{\partial\theta}{\partial\zeta} = \delta \frac{\partial^2\theta}{\partial\psi^2} \tag{19}$$

If

$$\eta(\zeta, \psi)_{\zeta=0} = \eta_0(\psi) \tag{20}$$

it may be assumed that

$$\theta(\zeta, \psi)_{\zeta=0} = \theta_0(\psi) = \exp\left\{-\frac{1}{2\delta}\int_0^\psi \eta_0(\psi)\, d\psi\right\} \tag{21}$$

and the general solution of equation (19) for the boundary condition (20) may be written in the following form:

$$\theta(\zeta, \psi) = \frac{1}{2\sqrt{\pi\delta\zeta}} \int_{-\infty}^{+\infty} \theta_0(\varphi) \exp\left(-\frac{(\psi-\varphi)^2}{4\delta\zeta}\right) \alpha\varphi \tag{22}$$

Let us assume [1,2] that the solution for η is a linear approximation having the form $\eta = -\eta_0 \sin\psi$, then the boundary condition for $\zeta = 0$ will be

$$\theta(\zeta, \psi)_{\zeta=0} = \exp\left(\frac{\eta_0}{2\delta}\cos\psi\right)$$

The solution of equation (19) for this boundary condition will have the form

$$\theta(\zeta, \psi) = \frac{1}{\sqrt{\pi\delta\zeta}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(\psi-\varphi)^2}{4\delta\zeta} + \frac{\eta_0\cos\varphi}{2\delta}\right] d\varphi = \sum_{n=0}^{\infty} A_n\, e^{-n^2\delta\zeta} \cos n\psi \tag{23}$$

$$\left(A_0 = I_0\left(\frac{\eta_0}{2\delta}\right), A_{n\neq 0} = 2I_n\left(\frac{\eta_0}{2\delta}\right)\right)$$

Here I_n is the modified (or hyperbolic) Bessel function. The rather unwieldy expression (23) is substantially simplified in the case of large and small conductivities of the medium. In the first case, the solution has the form of a progressive Riemann wave

$$\eta = -\eta_0 \sin(\psi - \zeta\eta) \tag{24}$$

which in an explicit form describes the nonlinear deformation of the wavefront. From equation (24) it follows that the effect of nonlinearity leads to a discontinuity of the wavefront at $\zeta \sim 1$.

In another limiting case with a small electric conductivity (with a large dispersion), solution (23) may be written in the form of a Taylor series, with respect to an infinitesimal parameter $\eta_0/2\delta$ which is correct to within $\eta_0/2\delta$ and has the following form:

$$\eta_0 = -\eta_0 \left\{ e^{-\delta\zeta} \sin\psi - \frac{\eta_0}{4\delta} (e^{-2\delta\zeta} - e^{-4\delta\zeta}) \sin 2\psi \right\} \tag{25}$$

In this case, the effect of nonlinearity of the equation of motion leads merely to the excitation of higher harmonics rather than causing a discontinuity of the wavefront. Formula (25) gives the dependence of the amplitude of the second harmonic with respect to the distance from the source.

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